Jurnal Matematika MANTIK
Vol. 8, No. 2, November 2022, pp.89-98

ISSN: 2527-3159 (*print*) 2527-3167 (*online*)

Mixed Boundary Value Problem for Nonlinear Fractional Volterra Integral Equation

Faraj Y. Ishak
University of Duhok Iraq, *faraj.ishak@uod.ac*

Article history:
Received Sep 9, 2022
Revised, Dec 9, 2022
Accepted, Dec 31, 2022

Kata Kunci:
Persamaan Volterra, teorema titik tetap Krasnoselskii, prinsip kontraksi Banach, teori *Leray-Schauder degree*.

Abstrak. Artikel ini menyajikan hasil penelitian tentang eksistensi dari solusi persamaan integral fraksional nonlinier tipe Volterra dengan kondisi batas campuran, beberapa hipotesis yang diperlukan telah dikembangkan untuk membuktikan keberadaan solusi persamaan yang diusulkan. Teorema Krasnoselskii, prinsip *Banach Contraction* dan teori derajat Leray-Schauder adalah teorema dasar yang digunakan di sini untuk mencari solusi hasil. Dalam artikel ini juga diberikan contoh sederhana dari penerapan hasil persamaannya.

Keywords:
Volterra equation, Krasnoselskii theorem, Banach contraction principle, Leray-Schauder degree theory.

Abstract. In this paper we present the existence of solutions for a nonlinear fractional integral equation of Volterra type with mixed boundary conditions, some necessary hypotheses have been developed to prove the existence of solutions to the proposed equation. Krasnoselskii Theorem, Banach Contraction principle and Leray-Schauder degree theory are the basic theorems used here to find the results. A simple example of application of the main result is presented.

How to cite:
F. Y. Ishak, "Mixed Boundary Value Problem for Nonlinear Fractional Volterra Integral Equation", *J. Mat. Mantik*, vol. 8, no. 2, pp. 89-98, December 2022.

CONTACT:
Faraj Y. Ishak faraj.ishak@uod.ac Department of Statistics, Collage of Administration & Economics University of Duhok, Iraq

The article can be accessed here. *https://doi.org/10.15642/mantik.2022.8.2.89-98*

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic. Recently, fractional differential equations have been of great interest. This is because of both the intensive development of the theory of fractional calculus itself and its applications in various sciences, such as physics, mechanics, chemistry, engineering.[1],[2],[3].

Integral equations appear in many engineering and scientific fields, such as unsteady aerodynamics, viscoelasticity, fluid dynamics, lots of population growth systems, neural network analysis, mathematical analysis of particle diffusion in an unstable fluid, heat conduction in memory resources, transmission lines, Population dynamics theory, nuclear reactors, inheritance systems. For details, see [4], [5], [6], [7], [8].

Boundary value problems have different applications. In addition to the previously mentioned fields, this type of problem appears in chemical engineering sciences, models of electromagnetic systems and thermoelectric theory. For more detailed information on boundary conditions, see [9], [10]. For more details on local and non-local boundary conditions, see [11],[12],[13],[14],[15],[16].

Feng, Zhang and Yang [13] in 2011 studied the existence and multiplicity solution to the nonlocal boundary value problem, fixed-point theorems in the cone were the main tool to prove the solutions. In 2014 Nyamoradi and Alaei [17] employed the Guo – Krasnoselskii fixed point theorem in a cone to study the existence of solution to a new fractional nonlocal mixed boundary value problem. In 2022 Ishak [18] investigated the existence solution for a fractional BVP of the first sort with Hadamard type and three-point boundary conditions using Krasnoselskii Zabriko theorem and Banach contraction principle.

It is also well known that fixed-point theorems have been applied to various boundary value problems to show the existence of solutions; for example, see [3],[5]. However, this researcher's remains not enough compared to the broad applications of this type of equations. The aim of this paper is to fill this gap. in this study we will investigate the existence and uniqueness solutions for the boundary value problem:

$$\boldsymbol{{}^{c}D^{\alpha}x(t) = g(t) + \int_{-\infty}^{t} \psi(t,s)\phi\big(t,s,x(s)\big)ds}$$
$$x(0) = ax(\gamma) + b,\ \ x'(0) = \omega\,,\ \ 0 \le t \le 1\,,\ 1 < \alpha \le 2 \tag{1}$$

Where $\boldsymbol{{}^{c}D}$ indicates the Caputo fractional operator $\boldsymbol{\phi}$: $\mathbf{[0,1]x[0,1]xX \to X}$ is a given continuous function in Banach space $(\mathbf{X},\|.\|)$ and $\mathbf{C = C([0,1],X)}$ is Banach space of all continuous functions from $[0,1]\to$ $\mathbf{X}$ endowed with the norm denoted by $\|.\|$, $\boldsymbol{a}\,, \boldsymbol{b}, \boldsymbol{\omega}\ [\mathbf{0},+\infty), \boldsymbol{\psi}$: $\mathbf{[0,1]x[0,1] \to X}$ is a given kernel , $\boldsymbol{g}$: $\mathbf{[0,1] \to [0,+\infty)}$ is known continuous function.

2. Preliminaries

In this section we will mention some basic definitions in fractional calculus.

***Definition 2.1*:** [19] The fractional integral of order q is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s)ds, \qquad q > 0$$

provided the integral exists.

Definition 2.2: [19] The fractional derivative of order q is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-\ s)^{n-q-1} f(s)ds,$$

$n-1 < q \leq n,\ q > 0$, Provided the right-hand side is pointwise defined on (0, +∞).

Lemma 2.1: [18] For α, β > 0, then the following relation hold:

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha-1}, \beta > n \text{ and } D^\alpha t^k = 0,$$
$$k = 0,1,\dots,n-1$$

Lemma 2.2: [18] Let $\alpha > 0$, then the differential equation

$${}^cD_{0+}^\alpha x(t) = 0$$

Has a unique solution $x(t) = c_0 + c_1 t + \cdots c_{n-1} t^{n-1}, c_i \in R, i = 1,2,\dots,n,$ where $n-1 < \alpha \leq n$

In view of Lemma 2.2, it follows that

$$I^q\ {}^cD^q x(t) = x(t) + c_0 + c_1 t + \cdots c_{n-1} t^{n-1},$$
$$c_i \in R, i = 1,2,\dots,n$$

Theorem 2.1: [5] (Krasnoselskii fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) Ax + By ∈ M whenever x, y ∈ M (ii) A is compact and continuous (iii) B is a contraction mapping. Then there exists z ∈ M such that z = Az + Bz.

Theorem 2.2: (Arzela -Ascoli theorem) Let *Ω* be a compact Hausdorff metric space. Then $M \subset C(\Omega)$ is relatively compact ⟺ M is uniformly bounded and uniformly equicontinuous.

Lemma 2.3: Given $f \in C(0,1) \cap L(0,1)$, the unique solution of (1) is:

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a}{1-a}\int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a\omega\gamma+b}{1-a} + \omega t \qquad (2)$$

Where,

$$f(t) = g(t) + \int_{-\infty}^t \psi(t,s)\phi\big(t,s,x(s)\big)ds$$

Proof: In view of lemma (2.2) the fractional differential equation (1) is equivalent to the integral equation:

$$x(t) = I_{0+}^\alpha f(t) + c_0 + c_1 t$$
$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + c_0 + c_1 t$$

Where $c_0, c_1 \in R$. From the boundary conditions (1), we have $c_1 = \omega$ and

$$c_0 = \frac{a}{1-a}\int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a\omega\gamma+b}{1-a}$$

By substituting c_0, c_1 in $x(t)$, the proof will be completed.
We will need the following hypotheses:
(H1) for $K \in R^+$ the inequality holds:

$$\|\phi(t,s,x_2) - \phi(t,s,x_1)\| \leq K\|x_2 - x_1\|, \quad \text{for all } t,s \in [0,1], \ x_1, x_2 \in X$$

(H2) $\|\phi(t,s,x)\| \leq L$ for all $t,s,x \in$ [0,1]x[0,1]xX, $L \in R^+$ Further
$\|g(t)\| \leq \xi$, $\|\psi(t,s)\| \leq \delta e^{-\lambda(t-s)}$ for all $\xi, \delta, \lambda \in R^+$

3. Main Result

Proves of theorems of existence and uniqueness solution for equation (1) will be given in this section.

Theorem 3.1: Suppose that ϕ: [0,1]x[0,1]x X → X is continuous and fulfilled H1and H2, if:

$$\frac{a\delta\, K\gamma^{\alpha}}{\lambda(1-a)\Gamma(\alpha+1)} < 1 \tag{3}$$

Then equation (1) has at least one solution.
Proof: Let $\varphi_r = \{x \in C: \|x\| \leq r\}$ where:

$$\frac{a\gamma^{\alpha}(\lambda\xi + \delta L)}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{\lambda\xi + \delta L}{\lambda\Gamma(\alpha+1)} + \frac{a\omega\gamma + b}{1-a} + \omega \leq r$$

define tow mapping F, G on φ_r s.t.

$$(Fx)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$$

$$(Gx)(t) = \frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a\omega\gamma + b}{1-a} + \omega t$$

For $x, y \in \varphi_r$, by (H2) we obtain:

$$\|(Fx)(t) + (Gx)(t)\| \leq$$

$$\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|g(s)\|ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big)\right\| ds\right)ds +$$

$$\left|\frac{a}{1-a}\right| \int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)}\|g(s)\|ds + \frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big)\right\| ds\right)ds +$$

$$\left|\frac{a\omega\gamma+b}{1-a}\right| + |\omega t|$$

$$\leq \frac{\xi}{\Gamma(\alpha+1)} + \frac{\delta}{\lambda}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L ds + \frac{a\xi\gamma^{\alpha}}{(1-a)\Gamma(\alpha+1)} + \frac{a\delta}{\lambda(1-a)}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} L ds + \frac{a\omega\gamma+b}{1-a} + \omega t$$

$$\leq \frac{\xi}{\Gamma(\alpha+1)} + \frac{\delta L}{\lambda\Gamma(\alpha+1)} + \frac{a\xi\gamma^{\alpha}}{(1-a)\Gamma(\alpha+1)} + \frac{a\delta L\gamma^{\alpha}}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{a\omega\gamma+b}{1-a} + \omega$$

$$\leq \frac{a\gamma^{\alpha}(\lambda\xi+\delta L)}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{\lambda\xi+\delta L}{\lambda\Gamma(\alpha+1)} + \frac{a\omega\gamma+b}{1-a} + \omega \leq r$$

Which means $Fx + Gx \in \varphi_r$.

Since F is continuous because ϕ is continuous, we have to prove that F is compact. (Fx)(t) is uniformly bounded on φ_r, as:

$$\|(Fx)(t)\| \leq \frac{\lambda\xi \ + \delta L}{\lambda\Gamma(\alpha+1)}$$

Since ϕ is bounded on [0,1]x[0,1]xφ_r let:

$$\phi_{max} = \underbrace{sup}_{s,t,x(t)\in[0,1]\text{x}[0,1]\text{x}\varphi_r} \|\phi(t,s,x(t))\|$$

Then for $t_1, t_2 \in [0,1]$ we get

$$\|(Fx)(t_2) - (Fx)(t_1)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\|((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})f(s)ds\| + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\|(t_2-s)^{\alpha-1}f(s)ds\|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left\|((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})\left(g(s) + \int_{-\infty}^{t}\psi(t,s)\phi\big(t,s,x(s)\big)\,ds\right)\right\| ds \ + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left\|(t_2-s)^{\alpha-1}\,(g(s) + \int_{-\infty}^{t}\psi(t,s)\phi\big(t,s,x(s)\big)\,ds)\right\| ds$$

$$\leq \frac{\xi|(t_2-t_1)^{\alpha} - t_2^{\alpha}|}{\Gamma(\alpha+1)} + \frac{\delta\phi_{max}|(t_2-t_1)^{\alpha} - t_2^{\alpha}|}{\lambda\Gamma(\alpha+1)} + \frac{\xi t_1^{\alpha}}{\Gamma(\alpha+1)} + \frac{\delta\phi_{max}t_1^{\alpha}}{\lambda\Gamma(\alpha+1)} - \frac{\xi|(t_2-t_1)^{\alpha}|}{\Gamma(\alpha+1)} - \frac{\delta\phi_{max}|(t_2-t_1)^{\alpha}|}{\lambda\Gamma(\alpha+1)}$$

$$\leq (\frac{\xi}{\Gamma(\alpha+1)} + \frac{\delta\phi_{max}}{\lambda\Gamma(\alpha+1)})|t_1^{\alpha} - t_2^{\alpha}|$$

Which is independent of x there for $(Fx)(t\,)$ is relatively compact on φ_r ,by Arzela-Ascoli's theorem $(Fx)(t\,)$ is compact in φ_r.
For $x, y \in \varphi_r$ and $t \in [0,1]$, by H1 we have:

$$\|(Gx)(t) - (Gy)(t)\| \leq \frac{a}{1-a}\int_0^{\gamma}\frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)}(\int_{-\infty}^{t}\|\psi(t,s)\|\,\|\phi\big(t,s,x(s)\big)\|ds)ds - \frac{a}{1-a}\int_0^{\gamma}\frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)}(\int_{-\infty}^{t}\|\psi(t,s)\|\,\|\phi\big(t,s,y(s)\big)\|ds)ds$$

$$\leq \frac{a}{1-a}\int_0^{\gamma}\frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)}(\int_{-\infty}^{t}\delta e^{-\lambda(t-s)}\|\phi\big(t,s,x(s)\big) - \phi\big(t,s,y(s)\big)\|\,ds)ds$$

$$\leq \frac{a\delta}{\lambda(1-a)}\int_0^{\gamma}\frac{(\gamma-s)^{\alpha-1}K}{\Gamma(\alpha)}\|x-y\|ds$$

$$\leq \frac{a\delta\,K\gamma^{\alpha}}{\lambda(1-a)\Gamma(\alpha+1)}\|x-y\| \qquad (4)$$

It follows from (4) that $(Gx)(t)$ is contraction mapping, this completes the prove.

Theorem 3.2: Suppose that ϕ: [0,1]x[0,1]xX → X is continuous and fulfilled (H1). If:

$$\Omega = \frac{\delta K + a\delta K(\gamma^{\alpha} - 1)}{\lambda(1-a)\Gamma(\alpha+1)} < 1 \tag{5}$$

Then equation (1) has a unique solution.

Proof: Let $\Phi: C \to C$ is defined as:

$$(\Phi x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \frac{a\omega\gamma+b}{1-a} + \omega t, \qquad t \in [0,1]$$

let $sup_{t\in[0,1]}|\phi(t,s,0)| = M$ and choose:

$$r \geq \frac{\lambda\xi+\delta M}{(1-\Omega)\lambda\Gamma(\alpha+1)} + \frac{a\gamma^{\alpha}(\lambda\xi+\delta M)}{(1-\Omega)\lambda a\Gamma(\alpha+1)} + \frac{a\omega\gamma+b}{(1-\Omega)(1-a)} + \frac{\omega t}{(1-\Omega)} \tag{6}$$

It is claimed that $\Phi\varphi_r \subset \varphi_r$ where:

$$\varphi_r = \{x \in C: \|x\| \leq r\}$$

In fact, for $x \in \varphi_r$, by (4), (5) and H1 we obtain:

$$\|(\Phi x)(t)\| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s)\|ds$$

$$+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big)\right\|ds\right) ds + \frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} \|g(s)\|ds +$$

$$\frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big)\right\|ds\right) ds + \left|\frac{a\omega\gamma+b}{1-a}\right| + |\omega t|$$

$$\leq \frac{\xi}{\Gamma(\alpha+1)} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_{-\infty}^t \delta e^{-\lambda(t-s)} \left(\left\|\phi\big(t,s,x(s)\big) - \phi(t,s,0)\right\| +\right.$$

$$\|\phi(t,s,0)\|)ds\, ds + \frac{a\xi\gamma^{\alpha}}{(1-a)\Gamma(\alpha+1)} + \frac{a}{1-a}\int_0^{\gamma} \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} (\int_{-\infty}^t \delta e^{-\lambda(t-s)} \left\|\phi\big(t,s,x(s)\big) -\right.$$

$$\left.\phi(t,s,0)\right\| + \|\phi(t,s,0)\|ds)ds + \frac{a\omega\gamma+b}{1-a} + \omega t$$

$$\leq \frac{\xi}{\Gamma(\alpha+1)} + \frac{r\delta K}{\lambda\Gamma(\alpha+1)} + \frac{\delta M}{\lambda\Gamma(\alpha+1)} + \frac{a\xi\gamma^{\alpha}}{(1-a)\Gamma(\alpha+1)} + \frac{ra\delta K\gamma^{\alpha}}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{a\delta M\gamma^{\alpha}}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{a\omega\gamma+b}{1-a} + \omega t$$

$$\leq r\left(\frac{\delta K + a\delta K(\gamma^{\alpha}-1)}{\lambda(1-a)\Gamma(\alpha+1)}\right) + \frac{\lambda\xi+\delta M}{\lambda\Gamma(\alpha+1)} + \frac{a\gamma^{\alpha}(\lambda\xi+\delta M)}{\lambda(1-a)\Gamma(\alpha+1)} + \frac{a\omega\gamma+b}{1-a} + \omega t$$

$$\leq r\Omega + (1-\Omega)r = r$$

Now we have to prove that the function Φ is contraction. For $x, y \in C$ and $t \in [0,1]$, by (5) and H1 we have:

$$\|(\Phi x)(t) - (\Phi y)(t)\|$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big) - \phi\big(t,s,y(s)\big)\right\| ds \right) ds +$$

$$\frac{a}{1-a} \int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} \left(\int_{-\infty}^t \|\psi(t,s)\| \left\|\phi\big(t,s,x(s)\big) - \phi\big(t,s,y(s)\big)\right\| ds \right) ds$$

$$\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} K\|x-y\| \left(\int_{-\infty}^t \delta e^{-\lambda(t-s)}\, ds \right. + \frac{a}{1-a} \int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} K\|x-y\| \left(\int_{-\infty}^t \delta e^{-\lambda(t-s)}\, ds \right) ds$$

$$\leq \frac{\delta K\|x-y\|}{\lambda\Gamma(\alpha+1)} + \frac{a\delta\gamma^\alpha K\|x-y\|}{\lambda(1-a)\Gamma(\alpha+1)}$$

$$\leq \frac{\delta K + a\delta K(\gamma^\alpha - 1)}{\lambda(1-a)\Gamma(\alpha+1)} \|x-y\| \leq \Omega\|x-y\|$$

$\Omega < 1$ ensure that $(\Phi x)(t)$ is contractive. Therefor the conclusion of the theorem follows from the contraction mapping principle.

Theorem 3.3: Let $\phi: [0,T] \times [0,T] \times R \to R$, and let $k \in R$ such that $0 \leq k < \frac{1}{\Omega}$, where

$$\Omega = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{a\gamma^\alpha}{(1-a)\Gamma(\alpha+1)}$$

and M > 0 such that $|\phi(t,s,x(t))| \leq k|x| + M$ for all $t \in [0,T], x \in R$ then problem (1) has at least one solution.

Proof: Define an operator $\Psi: \Lambda \to \Lambda$ as:

$$\Psi(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \frac{a}{1-a} \int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \frac{a\omega\gamma + b}{1-a} + \omega t$$

Where $\Lambda = C([0,1], R)$ denote to the Bansch space of all continuous functions from $[0,1] \to R$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. Let us define a fixed-point problem by:

$$x = \Psi x \tag{7}$$

Now we need to prove the existence of at least one solution $x \in \{0,T]$ satisfying (7). Define a ball $\mathcal{B}_r \subset C[0,T]$ with $r > 0$ as:

$$\mathcal{B}_r = \{x \in C[0,T]: \max_{t\in[0,T]} |x(t)| < r\}$$

Where r well be given later, then it's enough to show that $\overline{\mathcal{B}_r} \to C[0,T]$satisfies:

$$x \neq \sigma\Psi x, \forall x \in \partial\mathcal{B}_r \ and \ \forall\sigma \in [0,T] \tag{8}$$

Let us define $H(\sigma,x) = \sigma\Psi x, x \in C(\mathbb{R}), \sigma \in [0,T]$ Then by Arzesla'-Ascoli theorem $h_\sigma(x) = x - H(\sigma,x) = x - \sigma\Psi x$ is completely continuous if (8) is true then the following

Leray-Schauder degree are well define and by the homotopy invariance of topological degree it follows that:

$\deg(h_\sigma, \mathcal{B}_r, 0) = \deg(I - \sigma\Psi x, \mathcal{B}_r, 0) = \deg(h_1, \mathcal{B}_r, 0) = \deg(h_0, \mathcal{B}_r, 0) = \deg(I, \mathcal{B}_r, 0) = 1 \neq 0, 0 \in \mathcal{B}_r$. Where I denote the unit operator, by non-zero property of the Leray-Schauder degree $h_1(x) = x - \sigma\Psi x = 0$ for at least one $x \in \mathcal{B}_r$ in order to prove (8) we assume that $x = \sigma\Psi x$ for some $\sigma \in [0, T]$and for all $t \in [0, T]$ such that:

$$
\begin{aligned}
|x(t)| = |\sigma\Psi x| = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|g(t)| + \int_{-\infty}^t |\psi(t,s)||\phi(t,s,x(s))|ds)ds \\
&+ \frac{a}{1-a}\int_0^\gamma \frac{(\gamma-s)^{\alpha-1}}{\Gamma(\alpha)} (|g(t)| + \int_{-\infty}^t |\psi(t,s)||\phi(t,s,x(s))|ds)ds \\
&+ \left|\frac{a\omega\gamma + b}{1-a}\right| + |\omega t| \\
&\leq (\xi + \frac{\delta(k|x| + M}{\lambda})(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{a\gamma^\alpha}{(1-a)\Gamma(\alpha+1)}) + M_1 \\
&\leq (\xi + \frac{\delta(k|x| + M}{\lambda})\Omega + M_1
\end{aligned}
$$

Which on taking norm ($sup_{t\in[0,T]}|x| = \|x\|$) and solving for $\|x\|$ yields:

$$\|x\| = \frac{\xi\Omega(M\sigma + 1) + M_1}{\sigma(\sigma - \Omega\delta k)}$$

Where $M_1 = \left|\frac{a\omega\gamma+b}{1-a}\right| + |\omega t|$, letting

$$r = \frac{\xi\Omega(M\sigma + 1) + M_1}{\sigma(\sigma - \Omega\delta k)} + 1$$

(8) hold, this completes the proof.

Example: Consider the following fractional integrodifferential equation

$$
\begin{aligned}
D^{3/2}x(t) &= \sin(t) + \int_{-\infty}^t (2t+s)(t + x(s))ds \\
x(0) &= 0.5x(1.25) + 3, \quad \acute{x}(0) = 1.5 \qquad (9)
\end{aligned}
$$

Comparing (9) and (1), we see that $\alpha = 3/2$, $g(t) = \sin(t)$, $\psi(t,s) = 2t + s, \phi(t,s,x(t)) = t + x(s),\ \gamma = 1.25,\ a = 0.5,\ b = 3,\ \omega = 1.5.$ If we choose $\xi = 1$, $\delta = 1,\ \lambda = 5$, $K = 2$, then H1 holds, and

$$\frac{\delta K + a\delta K(\gamma^\alpha - 1)}{\lambda(1-a)\Gamma(\alpha+1)} = \frac{(1)(2) + (0.5)(1)(2)(1.25^{3/2} - 1)}{(5)(1-0.5)(\,1.3293)} < 1$$

That is, (5) holds. Thus, by theorem 3.2, equation (9) has a unique solution.

4. Conclusion

Based on the results and discussion, it can be concluded that the idea of Krasnoselskii fixed point theorem was very effective to proof the existence solution for the proposed equation. Also, under some suitable hypotheses and conditions we were able to complete the proof of existence solution for the equation proposed in this paper smoothly. The present work can be extended to the nonlocal and non-separable fractional boundary value problem.

References

[1] B. Ahmad, A. Alsaedi and B.S. Alghamdi, "Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions," *Nonlinear Anal Real World Appl*, vol. 9, no. 4, pp. 1727-1740, 2008.

[2] B. Ahmad, and A. Alsaedi, "Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions," *Nonlinear Anal Real World Appl*, vol. 10, no. 1, pp 358-367, 2009.

[3] B. Ahmad, and S. Sundaram, "Some existence results for fractional integro-differential equations with nonlinear conditions," *Communications in Applied Analysis*, vol. 12, no. 2, pp 107, 2008.

[4] B. Ahmad and B. S. Alghamdi, "Approximation of solutions of the nonlinear Duffing equation involving both integral and non-integral forcing terms with separated boundary conditions," *Comput Phys Commun*, vol. 179, no. 6, pp 409-416, 2008.

[5] B. Ahmad, J. J. Nieto, "Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions," *Bound Value Probl*, vol. 2009, pp 1-11.

[6] Mesloub and Said, "On a mixed nonlinear one point boundary value problem for an integrodifferential equation," *Bound Value Probl*, vol. 2008, pp 1-8.

[7] Y. K. Chang, A. Anguraj, and K. Karthikeyan, "Existence for impulsive neutral integrodifferential inclusions with nonlocal initial conditions via fractional operators," *Nonlinear Anal Theory Methods Appl*, vol. 71, no. 10, pp 4377-4386, 2009.

[8] Y. K. Chang, J.J. Nieto and W.S. Li, "Controllability of semi linear differential systems with nonlocal initial conditions in Banach spaces," *J Optim Theory Appl*, vol. 142, no. 2, pp 267-273, 2009.

[9] B. Ahmad, "On the existence of T-periodic solutions for Duffing type integro-differential equations with p-Laplacian," *LJM*, vol. 29, no. 1, pp 1-4, 2008.

[10] J.J. Nieto, Rodriguez-Lopez and Rosana, "New comparison results for impulsive integrodifferential equations and applications," *IMA J Numer Anal*, vol. 328, no. 2,

pp 1343-1368, 2007.

[11] A. Boucherif and Abdelkader, “Second-order boundary value problems with integral boundary conditions,” *Nonlinear Anal Theory Methods Appl*, vol. 70, no. 1, pp 346-371, 2009.

[12] M. Benchohra, S. Hamani and J. J. Nieto, “The method of upper and lower solutions for second order differential inclusions with integral boundary conditions,” *Rocky Mt J Math*, pp 13-26, 2010.

[13] M. Feng, X. Zhang and X. Yang, “Positive solutions of the order nonlinear impulsive differential equation with nonlocal boundary conditions,” *Bound Value Probl*, vol. 2011, pp 1-19.

[14] Y.K. Chang, J. J. Nieto and W. S. Li, “On impulsive hyperbolic differential inclusions with nonlocal initial conditions,” *J Optim Theory Appl*, vol. 140, no. 3, pp 431, 2009.

[15] Z. Luo and J. J. Nieto, “New results for the periodic boundary value problem for impulsive integrodifferential equations,” *Nonlinear Analysis, Theory, Methods & Applications*, vol. 70, no. 6, pp 2248-2260, 2009.

[16] Z. Yang, “Existence of nontrivial solutions for a nonlinear Sturm–Liouville problem with integral boundary conditions, *Nonlinear Analysis, Theory, Methods & Applications*, vol. 68, no. 1, pp 216-225, 2008.

[17] N. Nyamoradi and D. Alaei, “Existence solutions for nonlocal fractional differential equation with nonlinear boundary conditions,” *Iranian Journal of Science and Technology (Sciences)*, vol. 38, no. 4, pp 455-461, 2014.

[18] F. Y. Ishak, “Existence and uniqueness solution for three-point Hadamard type fractional Volterra BVP,” *E-Jurnal Matematika*, vol. 11, no.1, pp 70-78, 2022.

[19] Z. Wang, “Existence and Uniqueness of Solutions for a Nonlinear Fractional Integrodifferential Equation with Three-Point Fractional Boundary Conditions,” vol. 1, 2016.