Jurnal Matematika MANTIK
Vol. 7, No. 2, October 2021, pp. 107-114

ISSN: 2527-3159 (*print*) 2527-3167 (*online*)

Michaelis-Menten Models with Constant Harvesting of Restricted Prey Populations Minimum Place and Amount Capacity

Aswar Anas[1], Marsidi[2*]

[1,2]Department of Mathematics Education, Universitas PGRI Argopuro, Jember, Indonesia

Article history:
Received Jun 16, 2021
Revised, Jul 14, 2021
Accepted, Oct 31, 2021

Kata Kunci:
Model rantai makanan, Model Michaelis-Menten, Model mangsa dan pemangsa

Abstrak. Pemodelan rantai makanan saat ini sedang berkembang pesat. Ekosistem terlindungi dari rantai proses makan dan memakan. Semua makhluk hidup saling membutuhkan, namun jika proses memakannya tidak seimbang, maka kepunahan makhluk hidup akan terjadi. Salah satunya adalah model mangsa dan pemangsa yang berfungsi sebagai penyeimbang dalam sistem rantai makanan. Model Michaelis-Menten merupakan model mangsa-pemangsa yang pada intinya menjaga kepunahan mangsa. Permasalahannya adalah bagaimana menjaga mangsa tidak punah namun dengan pemanenan maksimal di suatu tempat dan jumlah minimum mangsa dengan waktu yang tepat. Metode yang digunakan untuk mengatasi masalah tersebut adalah menambah dua variabel baru pada model Michaelis-Menten, yaitu jumlah minimum mangsa dan kapasitas tempat yang akan ditempati. Terlihat bahwa sistem akan berada dalam keseimbangan jika tingkat kematian pemangsa besar, sehingga mangsa terjaga dari kepunahan sampai pemanenan. Selain itu waktu yang tepat untuk perkembangbiakan yang baik juga dapat ditentukan. Dari model ini didapatkan waktu yang tepat untuk pemanenan agar tidak terjadi kepunahan mangsa adalah $h = n\left(\frac{K^2-m^2}{4\,K}\right)$.

Keywords:
Food chain model, Michaelis-Menten model, Prey-predator model

Abstract. Food chain modeling is currently developing rapidly. The ecosystem is protected from the chain of eating and eating processes. All living things need each other, but if the process of eating them is not balanced, then the extinction of living things will occur. One of them is the prey and predator model that serves as a balancer in the food chain system. The Michaelis-Menten model is a prey-predator model that essentially prevents prey extinction. The problem is how to keep the prey from becoming extinct but with maximum harvesting in one place and the minimum amount of prey at the right time. The method used to overcome this problem is to add two new variables to the Michaelis-Menten model, namely the minimum number of prey and the capacity of the place to be occupied. It is seen that the system will be in equilibrium if the predator mortality rate is large so that the prey is kept from extinction until harvesting. In addition, the right time for good breeding can also be determined. From this model, it is found that the right time for harvesting so that prey extinction does not occur is $h = n\left(\frac{K^2-m^2}{4\,K}\right)$.

How to cite:
A. Anas and Marsidi, *"Michaelis-Menten Models with Constant Harvesting of Restricted Prey Populations Minimum Place and Amount Capacity", J. Mat. Mantik,* vol. 7*, no. 2, pp. 107-114, October 2021.*

CONTACT:
Marsidi, *marsidiarin@gmail.com* Departement of Mathematics Education Universitas PGRI Argopuro Jember, Jember, Jawa Timur 68121, Indonesia

The article can be accessed here. *https://doi.org/10.15642/mantik.2021.7.2.107-114*

1. Introduction

The food chain is a system created in nature. The dependence of living things on other living things is a normal interaction on this earth. This interaction occurs because of the needs of one party or two parties. The relationship between prey and predators is a one-sided relationship that harms one party, this is very closely related because predators can only survive if there is prey. As a result, the chances of predators experiencing extinction are small. In addition, the predator also functions as a controller of the growth rate of the prey [1].

Dongmei et al. stated that population extinction occurred because the initial population was too low [2]. If this happens, the prey population will be decrease and extinction may occur. As a result, predator populations are increasingly threatened indirectly. With the extinction of the Prey, the Predator also experiences extinction. The prey in this model is harvested while the predator population is not harvested because it has no commercial value. They are states that the commercial value is determined by the maximum value of h because if the prey harvested exceeds the value of h, the prey will become extinct.

The cause of the extinction of the Prey is also influenced by over-harvesting. Therefore, it is necessary to limit the number of prey so that the number of prey remains under control and does not exceed the existing capacity [3] .

The Lotka–Volterra model is frequently used to describe the dynamics of ecological systems in which two species interact, one a predator and one its prey. The model is simplified with the following assumptions: (1) only two species exist: fox and rabbit; (2) rabbits are born and then die through predation or inherent death; (3) foxes are born and their birth rate is positively affected by the rate of predation, and they die naturally [4]. The Michaelis-Menten model is a predator-prey model which is a generalization of the Lotka-Volterra model [5]. This model is used for events if the interaction system between individuals in a population has limited capacity. However, in this case, harvesting is also applied. Previously, researchers researched the Lotka-Volterra Proportional Predation and Prey model on the Von Bertallanfy [6] logistics modifications model . In this model, the researcher concludes that the number of predators in a place is highly dependent on the initial number of prey, the birth rate of the predator, and the birth rate of the prey. If the initial number of prey is large and the birth rate of prey is also large, the growth rate of predators increases. And if the birth constant of predators is small and the number of predators is smaller than the prey, there will be a decrease in the growth rate of prey. From previous research, researchers are interested in researching the Michaelis-Menten model, because this model prevents the extinction of both prey and predators. However, the researchers added the variables of space capacity and the minimum amount of prey that must be given.

2. Research Method

This research method is a literature study of the addition of the [7] variable with the capacity of the place and the minimum number of prey, the equation comes from:

According to [7], the equation for population growth and space capacity is formed by the formula:

$$\frac{dP(t)}{dt} = nP(t)\left(1 - \frac{P(t)}{K}\right) \quad (1)$$

From [8] with the addition of the assumption that the minimum number of population also affects the rate of population growth, then the form is:

$$\frac{dP(t)}{dt} = nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) \quad (2)$$

where n is the coefficient of the growth rate of the prey population, $P(t)$ is the number of prey populations at t, K is the capacity of the place, and m is the minimum number of populations.

Equation (2) is a logistic model that is influenced by the capacity of the place and the minimum number population. If the growth rate is also affected by the number of harvests of h, then equation (2) changes to:

$$\frac{dP(t)}{dt} = nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) - h \tag{3}$$

Where $0 \leq h \leq h_{max}$, h is the prey harvest rate constant and is the maximum harvested prey.

Xiao, et al describe the general formula for the persistence model in prey as follows [9]:

$$\begin{aligned} \frac{dP(t)}{dt} &= nP(t)\left(1 - \frac{P(t)}{K}\right) - \frac{cP(t)y(t)}{ry(t)+P(t)} - h \\ \frac{dy(t)}{dt} &= y(t)\left(-D + \frac{fP(t)}{ry(t)+P(t)}\right) \end{aligned} \tag{4}$$

where r is the satisfaction level of the Predator, c the number of prey captured, f the cconversion factor between the number of predators born for each prey captured, D is the mortality rate of the predator.

By adding the factor of the minimum number of prey population to reproduce, it is obtained

$$\begin{aligned} \frac{dP(t)}{dt} &= nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) - \frac{cP(t)y(t)}{ry(t)+P(t)} - h \\ \frac{dy(t)}{dt} &= y(t)\left(-D + \frac{fP(t)}{ry(t)+P(t)}\right) \end{aligned} \tag{5}$$

By [10] Equation (5) is sought a solution so that the prey does not experience extinction with:

a. Look for the maximum harvest value if predator is not present.
b. Search for fixed points.
c. Analysis of system stability at points T_1 and T_2.

3. Discussion

The following will discuss certain limits of harvesting a population to prevent extinction, then look for a fixed point from the Michaelis-Menten model to analyze the stability of the system at each of these fixed points and also perform simulations with different parameters.

3.1. Finding of Harvesting Maximum Value (h_{max}) Without Predator Condition

In equation (2) we have maximum points if:

$$\frac{dP(t)}{dt} = nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) = 0 \tag{6}$$

By [11] with a condition half of the carrying capacity and the minimum number of prey population $P(t)$. In this case, fixed point occurs in $P(t) = m$ and $P(t) = K$. If $P(t) = m$ then this point is not stable. It is caused by the population $P(t) > m$. It means that the population will grow rapidly and avoid the $P(t) = m$ and will attain $P(t) = K$. Let $m \leq P_0 < \left(\frac{K+m}{2}\right)$, point P_0 moves rapidly towards the maximum point when $P_0 = \left(\frac{K+m}{2}\right)$.

Whereas if $\left(\frac{K+m}{2}\right) < P_0 \leq K$, then this point moves slowly towards the stable point by K. If $0 \leq P_0 < m$, then the population will be moving down so that the population experiences extinction. While if $K < P_0$, then this point will be moving slowly down to stable point K.

Since harvesting results must be maximum, then the predator must be eliminated or equal to zero and $P(t) = \frac{K+m}{2}$, such that we have

$$\frac{dP(t)}{dt} = nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) - h = 0$$
$$\Rightarrow nP(t)\left(1 - \frac{P(t)}{K}\right)\left(1 - \frac{m}{P(t)}\right) = h$$
$$\Rightarrow \frac{n\big(K-P(t)\big)(P(t)-m)}{K} = h\ ;\ P(t) = \frac{K+m}{2}$$

So that:

$$n\left(\frac{K+m}{2}\right)\left(1 - \frac{K+m}{2K}\right)\left(1 - \frac{2m}{(K+m)}\right) = h$$

$$\Rightarrow n\left(\frac{K+m}{2}\right)\left(\frac{2K-K+m}{2K}\right)\left(\frac{K+m-2m}{K+m}\right) = h$$

$$\Rightarrow \frac{1}{2}n\left(\frac{K+m}{2K}\right)(K-m) = h$$

$$\Rightarrow n\left(\frac{K^2-m^2}{4K}\right) = h$$

Therefore, the maximum harvesting is bounded by $h = n\left(\frac{K^2-m^2}{4K}\right)$.

3.2. Finding Fixed Point

Let

$$f_1(P,y) = np\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right) - \frac{cPy}{ry+P} - h$$
$$f_2(P,y) = y\left(-D + \frac{fp}{ry+P}\right) \tag{7}$$

Suppose $f_1(P,y) = 0;\ f_2(P,y) = 0$ such that we have $y = 0$ or $\left(-D + \frac{fp}{ry+P}\right) = 0$.

$$\left(-D + \frac{fp}{ry+P}\right) = 0$$
$$\Leftrightarrow \frac{fp}{ry+P} = D$$
$$\Leftrightarrow fP = Dry + PD$$
$$\Leftrightarrow \frac{fP-PD}{Dr} = y$$

Then we have $y_1 = 0;\ y_2 = \frac{P(f-D)}{Dr}$. Furthermore, finding the points P_1 and P_2 by substituting $y_1 = 0$ to $f_1(P, y_1) = 0$.

$$f_1(P,y) = np\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right) - \frac{cP0}{r0+P} - h = 0$$
$$\Leftrightarrow np\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right) - h = 0$$

$\Leftrightarrow np\left(\frac{K-P}{K}\right)\left(\frac{p-m}{P}\right)-h=0$
$\Leftrightarrow \frac{n}{K}(KP-Km+Pm-P^2)-h=0$
$\Leftrightarrow nP-nm+\frac{P}{K}mn-\frac{n}{K}P^2-h=0$
$\Leftrightarrow -\frac{n}{K}P^2+P\left(n+\frac{mn}{K}\right)-(h+nm)=0$
$\Leftrightarrow \frac{n}{K}P^2-\left(n+\frac{mn}{K}\right)P+(h+nm)=0$
Such that

$$P_{1,2}=\frac{\left(n+\frac{mn}{K}\right)\pm\sqrt{\left(-\left(n+\frac{mn}{K}\right)\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)}$$
$$P_{1,2}=\frac{\left(n+\frac{mn}{K}\right)\pm\sqrt{\left(n+\frac{mn}{K}\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)}$$

For $y_2=\frac{P(f-D)}{Dr}$ we have

$np\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)-\left(\frac{cP\frac{P(f-D)}{Dr}}{r\frac{P(f-D)}{Dr}+P}\right)-h=0$
$\Leftrightarrow np\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)-\left(\frac{cP\frac{P(f-D)}{Dr}}{r\frac{P(f-D)+PDr}{Dr}}\right)-h=0$
$\Leftrightarrow np\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)-\frac{(cP^2f-cP^2D)}{rPf-PDr+PDr}-h=0$
$\Leftrightarrow np\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)-\frac{(cP^2f-cP^2D)}{rPf}-h=0$
$\Leftrightarrow \frac{n}{K}(KP-Km+Pm-P^2)-\frac{(cP^2f-cP^2D)}{rPf}-h=0$
$\Leftrightarrow \frac{rPfn}{K}(KP-Km+Pm-P^2)-(cP^2f-cP^2D)-hrPf=0$
$\Leftrightarrow P\left(\frac{rfn}{K}(KP-Km+Pm-P^2)-(cPf-cPD)-hrf\right)=0$
$\Leftrightarrow P\left(\left(rfnP-rfnm+\frac{rfnPm}{K}-\frac{rfnP^2}{K}\right)-cPf+cPD=hrf\right)=0$
$\Leftrightarrow P\left(-\frac{rfn}{K}P^2+P\left(rfn+\frac{rfnm}{K}-cf+cD\right)-(hrf+rnfm)\right)=0$

$\Leftrightarrow P\left(\frac{rfn}{K}P^2-\left(rfn+\frac{rfnm}{K}-cf+cD\right)P+(hrf+rnfm)\right)=0$

Thus $P=0$ or
$$P_{1,2}=\frac{\left(rfn+\frac{rfnm}{K}-cf+cD\right)\pm\sqrt{\left(-\left(rfn+\frac{rfnm}{K}-cf+cD\right)\right)^2-4\left(\frac{rfn}{K}\right)(hrf+rnfm)}}{2\left(\frac{rfn}{K}\right)}$$
$$P_{1,2}=\frac{\left(rfn\frac{(K+m)}{K}-cf+cD\right)\pm\sqrt{\left(rfn+\frac{rfnm}{K}-cf+cD\right)^2-4\left(\frac{rfn}{K}\right)(hrf+rnfm)}}{2\left(\frac{rfn}{K}\right)}$$

It is impossible for $P=0$. Since this model must have a minimum amount so that the prey can grow.

From the results above, we have a fixed points as follows:

$$T_1:(P_1,y_1)=\left(\frac{\left(n+\frac{mn}{K}\right)+\sqrt{\left(\left(n+\frac{mn}{K}\right)\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)},0\right)$$

$$T_2:(P_2,y_1)=\left(\frac{\left(n+\frac{mn}{K}\right)-\sqrt{\left(n+\frac{mn}{K}\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)},0\right)$$

$$T_3:(P_1^*,y_2^*)=\left(\frac{\left(rfn\frac{(K+m)}{K}-cf+cD\right)+\sqrt{\left(rfn+\frac{rfnm}{K}-cf+cD\right)^2-4\left(\frac{rfn}{K}\right)(hrf+rnfm)}}{2\left(\frac{rfn}{K}\right)},\frac{P_1^*(f-D)}{Dr}\right)$$

$$T_4:(P_2^*,y_2^*)=\left(\frac{\left(rfn\frac{(K+m)}{K}-cf+cD\right)-\sqrt{\left(rfn+\frac{rfnm}{K}-cf+cD\right)^2-4\left(\frac{rfn}{K}\right)(hrf+rnfm)}}{2\left(\frac{rfn}{K}\right)},\frac{P_2^*(f-D)}{Dr}\right)$$

Equilibrium analysis is carried out to find out the points that cause the system to be in equilibrium and not, to analyze it, look for real eigenvalues at each equilibrium point [12]. To find the stability of the equilibrium point of a model, you can use the Jacobian matrix with the order 2×2. This matrix is discussed in the next sub-chapter.

3.3. Jacobian Matrix

In [13] let the equation system equation (5) written by:

$$\frac{dP}{dt}=f_1(P,y)$$
$$\frac{dY}{dt}=f_2(P,y)$$

Such that, the Jacobian matrix can be formed by:

$$J=\begin{bmatrix}\frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{P} & \frac{\partial f_2}{\partial y}\end{bmatrix}$$

$$J=\begin{bmatrix}-\frac{2np}{K}+\frac{n(K+m)}{K}-\frac{rcy^2+cyP+rcPy^2}{(ry+P)^2} & -\frac{cP^2}{(ry+P)^2}\\ \frac{fry^2}{(ry+P)^2} & -D+\frac{fry^2+fyP-fPyr}{(ry+P)^2}\end{bmatrix}$$

The stability of the system of equations (5) will be known by analyzing the eigen values of the Jacobian matrix.

3.4. Fixed Point Stable Analysis

The following discussion discusses the stability analysis of fixed points using the Jacobi matrix of order 2×2 and only discussed in point $T_1(P_{1,}y_1)$, $T_2(P_{2,}y_1)$.

3.4.1 Stable system at fixed point T_1

By [14] we take the fixed point

$T_1 : (P_1, y_1) = \left(\frac{\left(n+\frac{mn}{K}\right)+\sqrt{\left(n+\frac{mn}{K}\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)}, 0 \right)$. We substitute the point T_1 to Jacobian Matrix as follows.

$$J_1 = \begin{bmatrix} -\frac{2np}{K} + \frac{n(K+m)}{K} - \frac{rcy^2+cyP+rcPy^2}{(ry+P)^2} & -\frac{cP^2}{(ry+P)^2} \\ \frac{fry^2}{(ry+P)^2} & -D + \frac{fry^2+fyP-fPyr}{(ry+P)^2} \end{bmatrix}$$

Then we have:

$$J_1 = \begin{bmatrix} -\frac{2np}{K} + \frac{n(K+m)}{K} - \frac{rc0+c0P+rcP0}{(r0+P)^2} & -\frac{cP^2}{(r0+P)^2} \\ 0 & -D \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -\frac{2np}{K} + \frac{n(K+m)}{K} & -c \\ 0 & -D \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -n - \frac{mn}{K} - \sqrt{\left(n + \frac{mn}{K}\right) - \frac{4n(mn+h)}{K}} + \frac{n(K+m)}{m} & -c \\ 0 & -D \end{bmatrix}$$

$$J_1 = \begin{bmatrix} -\sqrt{\left(n + \frac{mn}{K}\right)^2 - \frac{4n(mn+h)}{K}} & -c \\ 0 & -D \end{bmatrix}$$

The eigen value can be obtained by $det(A - \lambda_1 I) = 0$.

$$\left(-\sqrt{\left(n + \frac{mn}{K}\right)^2 - \frac{4n(mn+h)}{K}} - \lambda\right)(-D - \lambda) = 0$$

$$\Leftrightarrow \lambda_1 = -\sqrt{\left(n + \frac{mn}{K}\right)^2 - \frac{4n(mn+h)}{K}} \quad \text{or} \quad \lambda_2 = -D$$

with $h = n\left(\frac{K^2-m^2}{4K}\right)$ we have $\lambda_1 = -\sqrt{2}\sqrt{-\frac{n^2 m(K-m)}{K^2}}$ or $\lambda_2 = -D$.

From the equation above we know that λ_1 is not real eigen value because n, m, K is a positive parameter with $K > m$ and $\lambda_2 \leq 0$ if D is a positive integer. It show that T_1 is stable.

3.4.2 Stable system at fixed point T_2

Let $T_2 : (P_2, y_1) = \left(\frac{\left(n+\frac{mn}{K}\right)-\sqrt{\left(n+\frac{mn}{K}\right)^2-4\left(\frac{n}{K}\right)(h+nm)}}{2\left(\frac{n}{K}\right)}, 0 \right)$. With the same way we obtain that $\lambda_1 = \sqrt{\left(n + \frac{mn}{K}\right)^2 - \frac{4n(mn+h)}{K}}$ or $\lambda_2 = -D$. Wit $h = n\left(\frac{K^2-m^2}{4K}\right)$ we get $\lambda_1 = \sqrt{2}\sqrt{-\frac{n^2 m(K-m)}{K^2}}$ or $\lambda_2 = -D$, because λ_1 is not real eigen value and $\lambda_2 \leq 0$ if D is a positive integer. It show that T_2 is stable [15].

4. Conclusion

From the calculation results obtained 4 fixed points. In here we are only discussing points that are $T_1$and T_2. The analysis carried out at two fixed points states that the equilibrium and stability of the population is influenced by the level of capacity, the minimum number of population, and the death rate of the predator. From the discussion section we know that maximum harvesting can be done if $h = n\left(\frac{K^2 - m^2}{4K}\right)$, it's aim is to ensure that prey and predator populations do not become extinct.

5. Acknowledgments

We gratefully acknowledge the support from Universitas PGRI Argopuro Jember 2021.

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