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Duflo-Moore Operator for The Square-Integrable Representation of the 2-Dimensional Affine Lie Group

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Abstrak. Dalam artikel ini, dipelajari representasi *quasi-regular* dan representasi unitar tak tereduksi grup Lie *affine* $\mathrm{Aff}^+(1)$ berdimensi dua. Pertama, diberikan bukti lengkap dari hasil kerja Fuhr tentang transformasi *Fourier* untuk representasi *quasi-regular* dari $\mathrm{Aff}^+(1)$. Kedua, ketika representasi dari grup Lie *affine* $\mathrm{Aff}^+(1)$ adalah *square-integrable* maka dihitung operator Duflo-Moore secara langsung tanpa menggunakan transformasi *Fourier* seperti dalam hasil Fuhr.

Kata kunci: Grup Lie affine; Operator Duflo-Moore; Representasi square-integrable.

Abstract. In this paper, we study the quasi-regular and the irreducible unitary representation of affine Lie group $\mathrm{Aff}^+(1)$ of dimension two. First, we prove a sharpening of Fuhr's work of Fourier transform of quasi-regular representation of $\mathrm{Aff}^+(1)$. The second, in such the representation of affine Lie group $\mathrm{Aff}^+(1)$ is square-integrable then we compute its Duflo-Moore operator instead of using Fourier transform as in Führ's work.

Keywords: Affine Lie group; Duflo-Moore operator; Square-integrable representation.

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1. Introduction

The current research about square-integrable representations of Lie groups can be found, for instance in [1] and [2]. In the previous work, the notion of square-integrable representation of a Lie group associating to wavelet transforms was introduced by Grossmann, Morlet, and Paul (see [3]). Particularly, they investigated the nice examples of a square-integrable representation of $ax+b$- group, known as affine Lie group $\mathrm{Aff}(1)$ as can be seen in [4]. In the other hand, the research about $ax+b$-groups can also be found, for instance in [5] and [6].

It is well known that $\mathrm{Aff}(1)$ is the exponential solvable Lie group which is non unimodular group whose Lie algebra of $\mathrm{Aff}(1)$ is Frobenius. Other examples are parabolic subgroups which are Fobenius as well (see [7] and[8]). But we thought that Grossmann's work is the best example for young researchers how to understand the square-integrable representations for case nonunimodular groups which is started from the $\mathrm{Aff}(1)$ Lie group. Moreover, other examples of nonunimodular groups are Lie groups whose Lie algebras are 4-dimensional real Frobenius Lie algebras. Kurniadi and Ishi [9] showed that irreducible unitary representations of these Lie groups are square-integrable representations and they wrote the Duflo-Moore operators in the terms of groups Fourier transforms.

Many reseachers study affine Lie algebras and the structure of affine for instance we see some results in [10], [11], [12], [13],[14], [15], [16], and [17].

In the other hand, in easier stage we can also study square-integrable representations for unimodular Lie groups case. Heisenberg Lie groups of dimension $2n+1$ and filiform Lie groups are in these types. In fact, the Duflo-Moore operators for square-integrable representations of unimodular groups are scalar multiple (see [18]). In current work, Kurniadi in [19] proved that irreducible-unitary representation of Lie group of 4-dimensional standard filiform Lie algebra is square-integrable and its Duflo-Moore operator is scalar multiple of identity which is equal to one.

In this work, we shall give another alternative to compute the Duflo-Moore operator for square-integrable representation of $\mathrm{Aff}^+(1)$ by direct computations instead of forming in group Fourier transform which was written in [18].

2. Preliminaries

Let $\mathrm{Aff}^+(1)$ be the 2-dimensional affine Lie group whis is expressed as a semidirect product of the set of all real numbers $\mathbb{R}$ and the set of all positive real numbers $\mathbb{R}_+$. Namely, we can write this group as $\mathrm{Aff}^+(1) \coloneqq \mathbb{R} \rtimes \mathbb{R}_+$. Particularly, in this work we concentrate to $\mathrm{Aff}^+(1)$ which is the exponential solvable nonunimodular Lie group. To make easier in computations we write $\mathrm{Aff}^+(1)$ in matrix terms. Namely, we have

$$\mathrm{Aff}^+(1) \ni \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix},\ \alpha \in \mathbb{R}_+,\ \beta \in \mathbb{R}. \tag{1}$$

Regarding this notations, we denote $g(\alpha,\beta) \coloneqq \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, $\Delta(\alpha) \coloneqq \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, and $\nabla(\beta) \coloneqq \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. The Lie algebra of $\mathrm{Aff}^+(1)$ is denoted by $\mathrm{aff}(1)$ whose basis is $\{e_1, e_2\}$. The nonzero bracket of $\mathrm{aff}(1)$ is given by $[e_1, e_2] = e_2$. The Lie algebra $\mathrm{aff}(1)$ is a Frobenius Lie algebra which has two open coadjoint orbits as follows (see [20]).

$$\Omega_{\pm} := \{(a,b) \quad ; \quad a, b \in \mathbb{R}, \quad \pm b > 0\}. \tag{2}$$

The representations of the affine Lie group $\mathrm{Aff}(1)$ can be realized on the Hilbert space of all square-integrable functions $\mathrm{L}^2(\mathbb{R}_+)$. Before doing that, let us mention here some basic notion of representation theory of Lie groups corresponding to our research.

Definition 1 [21]**.** Let π be a representation of a Lie group G on the carrier space $\mathcal{H}$. π is said to be irreducible if π has no nontrivial π-invariant subspace $\mathcal{H}_0$ in $\mathcal{H}$. Moreover, π is said to be uintary if for each $f \in \mathcal{H}$ and each $g \in G$

$$\|\pi(g)f\| = \|f\|. \tag{3}$$

Proposition 2 [20]**.** The irreducible unitary representations of $\mathrm{Aff}^+(1)$ correponsding to open coadjoint orbit Ω_+ in eqs. (2) in the space $\mathrm{L}^2(\mathbb{R}_+)$ is of the form

$$\pi_+(g)f(x) = e^{2\pi i\beta x}f(\alpha x), \tag{4}$$

where $g \coloneqq g(\alpha,\beta) \in \mathrm{Aff}^+(1)$, $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$ *and* $f \in \mathrm{L}^2(\mathbb{R}_+)$.

Furthermore, the representation of affine Lie group $\mathrm{Aff}^+(1)$ can be realized as a quasi-regular representations (see [18]). It is written in the formula as follows.

$$\pi(g(\alpha,\beta)) = \alpha^{-\frac{1}{2}}\psi(\frac{x-\beta}{\alpha}),\ \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \text{ and } \psi \in \mathrm{L}^2(\mathbb{R}_+). \tag{5}$$

We are mainly interested in the square-integrable representation. Let π be an irreducible unitary representation of a Lie group G realized on the space $\mathcal{H}$ and $\mathrm{L}^2(G)$ be the space of all square-integrable functions on G. For vector $f_1 \in \mathcal{H}$, we define the operator on $\mathcal{H}$ given by

$$\mathcal{E}_{f_1}\colon \mathcal{H} \ni f_2 \mapsto \mathcal{E}_{f_1}f_2 \in \mathrm{L}^2(G). \tag{6}$$

where $\mathcal{E}_{f_1}f_2(x) = \langle f_1|\pi(x)f_2\rangle$.

Definition 3 [22]**.**The irreducible unitary representation π of locally compact topological group G realized on a space $\mathcal{H}$ is said to be square-interable if there exist two vectors $f_1, f_2 \in \mathcal{H} - \{0\}$ such that

$$\left\|\mathcal{E}_{f_1}f_2\right\|^2 = \langle f_1|\pi(x)f_2\rangle = \int_G f_1(g)\overline{\pi(x)f_2(g)}\ \mathrm{d}\mu(g) < +\infty. \tag{7}$$

In the other words, $\langle f_1|\pi(x)f_2\rangle \in \mathrm{L}^2(G,\mu_G)$ where μ_G is a measure on G. Such vectors which satisfied eqs. (7) are called admissible vectors.

Duflo-Moore state their results in the following theorem

Theorem 4 [23]**.** If π is square-integrable representations of locally compact group G realized on the space $\mathcal{H}$ then there exists a positive selfadjoint operator $C_\pi\colon \mathcal{H} \to \mathcal{H}$ which is called **the Duflo-Moore operator** such that

a. a vector $\psi \in \mathcal{H} - \{0\}$ is admissible if and only if ψ is an element of domain of C_π.
b. if $f_1, f_2 \in \mathcal{H}$ and $f_3, f_4 \in \mathrm{Dom}\ (C_\pi)$ then

$$\langle \mathcal{E}_{f_1}f_3|\mathcal{E}_{f_2}f_4\rangle_{\mathrm{L}^2(G,\mu_G)} = \langle f_1|f_2\rangle_{\mathcal{H}}\langle C_\pi f_4|C_\pi f_3\rangle_{\mathcal{H}}. \tag{8}$$

2. Methods

In this research we apply the literature reviews method, particularly we focus on results in [18] and [20]. We obtain the quasi-regular representation of $\mathrm{Aff}^+(1)$ in Fuhr's work and we compute the Fourier transform of its representation to determine the Duflo-Moore operator. On the other hand, we also obtain the irreducible unitary representation of $\mathrm{Aff}^+(1)$ corressponding to open coadjoint orbits and we show that representatation is square-integrable. Using direct computations, we obtain the Duflo-Moore operator for that representation.

3. Results and Discussion

Our results and discussion consist of two main part as follows.

3.1 The Duflo-Moore Operator for The Quasi-Regular Representation of $\mathbf{Aff}^+(\mathbf{1})$.

The following statement can be deduced from [18] in page 30--31. However, we give a detail proof for its own interest.

Lemma 5 [18]. The Fourier transform of quasi-regular representation π of $\mathrm{Aff}^+(1)$ as in eqs. (5) is of the form

$$\mathcal{F}(\pi(g(\alpha,\beta))\psi)(\xi) = \alpha^{\frac{1}{2}} e^{-2\pi i\xi\beta}\mathcal{F}\psi(\alpha\xi). \tag{9}$$

Proof.

By direct computation we obtain

$$\begin{aligned}
\mathcal{F}(\pi(g(\alpha,\beta))\psi)(\xi) &= \int_{\mathbb{R}} e^{-2\pi i\xi x}(\pi(g(\alpha,\beta))\psi)(x)\ dx \\
&= \int_{\mathbb{R}} e^{-2\pi i\xi x}\alpha^{-1/2}\psi\left(\frac{x-\beta}{\alpha}\right)\ dx \\
&= \int_{\mathbb{R}} e^{-2\pi i\xi(\alpha\eta+\beta)}\alpha^{-1/2}\psi(\eta)\,\alpha \quad d\eta \\
&\qquad (\text{ Substituting } \eta = \tfrac{x-\beta}{\alpha}\ ) \\
&= \int_{\mathbb{R}} e^{-2\pi i\xi(\alpha\eta)} e^{-2\pi i\xi\beta}\alpha^{1/2}\psi(\eta)\ d\eta \\
&= \int_{\mathbb{R}} e^{-2\pi i(\alpha\xi)\eta} e^{-2\pi i\xi\beta}\alpha^{1/2}\psi(\eta)\ d\eta \\
&= e^{-2\pi i\xi\beta}\alpha^{1/2}\int_{\mathbb{R}} e^{-2\pi i(\alpha\xi)\eta}\psi(\eta)\ d\eta \\
&= e^{-2\pi i\xi\beta}\alpha^{\frac{1}{2}}\mathcal{F}\psi(\alpha\xi).
\end{aligned}$$

■

Proposition 6 [18]**.** The Duflo-Moore operator for quasi-regular representation π of $\mathrm{Aff}^+(1)$ as in eqs. (5) in the term of Fourier transform can be written as follows.

$$\mathcal{F}(C_\pi\psi)(\xi) = \xi^{-1/2}\mathcal{F}\psi(\xi). \tag{10}$$

Proof. Let ψ_1 and ψ_2 be elements of contiunuous functions space of compact support on $\mathrm{Aff}^+(1)$ denoted by $C_c(\mathrm{Aff}^+(1))$. Using Plancherel's theorem and Fubini's theorem we obtain

$$\int_{\mathrm{Aff}^+(1)} |\langle\psi_1|\pi(g(\alpha,\beta))\psi_2\rangle|^2 \frac{d\alpha}{\alpha^2}d\beta = \int_{\mathrm{Aff}^+(1)} |\langle\mathcal{F}\psi_1|\mathcal{F}\pi(g(\alpha,\beta))\psi_2\rangle|^2 \frac{d\alpha}{\alpha^2}d\beta$$

$$= \int_{\mathrm{Aff}^+(1)} \left|\int_{\mathbb{R}} \mathcal{F}\psi_1(\xi)\overline{e^{-2\pi\iota\xi\beta}\alpha^{1/2}\mathcal{F}\psi(\alpha\xi)}\ d\xi\right|^2 \frac{d\alpha}{\alpha^2}d\beta$$

$$= \int_{\mathrm{Aff}^+(1)} \left|\int_{\mathbb{R}} \mathcal{F}\psi_1(\xi)e^{2\pi i\xi\beta}\alpha^{1/2}\overline{\mathcal{F}\psi}(\alpha\xi)\ d\xi\right|^2 \frac{d\alpha}{\alpha^2}d\beta$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+} \left|\int_{\mathbb{R}} \mathcal{F}\psi_1(\xi)e^{2\pi i\xi\beta}\overline{\mathcal{F}\psi}(\alpha\xi)\ d\xi\right|^2 \frac{d\alpha}{\alpha}d\beta$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+} |\mathcal{F}\tau_\alpha(-\beta)|^2 \frac{d\alpha}{\alpha}d\beta$$

$$(\ \tau_\alpha(\xi) = \mathcal{F}\psi_1(\xi)\overline{\mathcal{F}\psi}(\alpha\xi)\ )$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_+} \left|\mathcal{F}\psi_1(\xi)\overline{\mathcal{F}\psi}(\alpha\xi)\right|^2 \frac{d\alpha}{\alpha}d\beta$$

$$= \int_{\mathbb{R}} |\mathcal{F}\psi_1(\xi)|^2 \left\{\int_{\mathbb{R}_+} \left|\overline{\mathcal{F}\psi}(\alpha\xi)\right|^2 \frac{d\alpha}{\alpha}\right\} d\xi$$

$$= \{\int_{\mathbb{R}} |\mathcal{F}\psi_1(\xi)|^2 d\xi\ \} \left\{\int_{\mathbb{R}_+} \left|\overline{\mathcal{F}\psi}(\alpha\xi)\right|^2 \frac{d\alpha}{\alpha}\right\}$$

$$= \{\int_{\mathbb{R}} |\mathcal{F}\psi_1(\xi)|^2 d\xi\ \} \left\{\int_{\mathbb{R}} \left|\overline{\mathcal{F}\psi}(\alpha')\right|^2 \frac{d\alpha'}{\alpha'}\right\}$$

$$(\ \alpha' \coloneqq \alpha\xi\ ).$$

$$= \|\mathcal{F}\psi_1\|^2 \left\{\int_{\mathbb{R}} \left|\overline{\mathcal{F}\psi}(\alpha')\right|^2 \frac{d\alpha'}{\alpha'}\right\}.$$

Thus, from the latter equation we obtain the Duflo-Moore operator is equal to $\mathcal{F}(C_\pi\psi)(\xi) = \xi^{-1/2}\mathcal{F}\psi(\xi)$ as desired.

■

3.2 The Duflo-Moore Operator for The Irreducible Unitary Representation of $\mathbf{Aff^+(1)}$

This session is the main result. First, we recall that the irreducible unitary representation of group $\mathrm{Aff}^+(1)$ in Proposition 2 can be written in the following proposition

Proposition 7. The irreducible unitary representations of $\mathrm{Aff}^+(1)$ correponsding to open coadjoint orbit Ω_+ in eqs. (2) in the space $\mathrm{L}^2(\mathbb{R}_+)$ is of the form

$$\pi_+(\Delta(\alpha))f(x) = f(\alpha x),$$
$$\pi_+(\nabla(\beta))f(x) = e^{2\pi i\beta x}f(x), \tag{11}$$

where $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$ and $f \in \mathrm{L}^2(\mathbb{R}_+)$.

Proof. Let $\mathrm{aff}(1)$ be a Lie algebra of $\mathrm{Aff}^+(1)$ whose basis is $\{e_1, e_2\}$. We consider its dual space as $\mathrm{aff}(1)^* \ni \begin{pmatrix} a & * \\ b & * \end{pmatrix}$, where $a, b \in \mathbb{R}$. Moreover, let $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ be an element of group affine $\mathrm{Aff}^+(1)$. We shall construct the irreducible unitary representation of $\mathrm{Aff}^+(1)$ corresponding to open coadjoint orbit $\Omega_+ \coloneqq \{(a, b) \quad ; \quad b > 0\}$. To do that, fix a point $\tau \coloneqq e_2^* \in \Omega_+ \subset \mathrm{aff}(1)^*$ as a linear functional. For subalgebra $\aleph \coloneqq \langle e_2 \rangle$ we have $\aleph$ has maximal dimension and the value of linear functional τ on the commutator $[\aleph, \aleph]$ is given by $\tau([\aleph, \aleph]) = 0$. Therefore, $\aleph$ is a polarization in $\mathrm{aff}(1)$. Let $\aleph^\perp$ be the orthogonal subspace. Furthermore, since $\tau + \aleph^\perp$ is contained in Ω_+ then $\aleph$ satisfies Pukanszky condition.

Now we construct a 1-dimensional representation λ_τ of $\mathrm{N} \coloneqq \exp \aleph$ as follows.

$$\lambda_\tau(\exp e) \coloneqq e^{2\pi i\langle\tau|e\rangle} = e^{2\pi i\beta}, e \coloneqq \alpha e_1 + \beta e_2, \tau \in \Omega_+. \tag{12}$$

We identify the coset $\mathrm{Aff}^+(1)/\mathrm{N}$ by $\mathbb{R}_+$ and we obtain the section given by

$$s: \mathbb{R}_+ \ni x \mapsto \exp xe_1 \in \mathrm{Aff}^+(1). \tag{13}$$

To obtain the explicit formula of the representation of $\mathrm{Aff}^+(1)$ we need to solve what we called the master equation

$$s(x)g = h_s(x, g)s(xg), \qquad (x \in \mathbb{R}_+, g \in \mathrm{Aff}^+(1), h_s(x, g) \in \mathrm{N}). \tag{14}$$

Using the basis $\{e_1, e_2\}$ we solve the following master equations with respect to its basis:

a. $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$,

by solving with respect to u and y we obtain $y = \alpha x$. Therefore, $\pi_+(\Delta(\alpha))f(x) = f(\alpha x)$. We mention here that we apply a right action of $\mathrm{Aff}^+(1)$ in space $\mathrm{L}^2(\mathbb{R}_+)$.

b. $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$.

In this case, we have $y = x$ and $u = \beta x$. Therefore, $\pi_+(\nabla(\beta))f(x) = e^{2\pi i\beta x}f(x)$ as desired.

■

In the next section, we shall compute the Duflo-Moore operator for the representation of $\mathrm{Aff}^+(1)$ with respect to its right Haar measure. The result of Duflo-Moore operator for the representation of $\mathrm{Aff}^+(1)$ with respect left Haar measure can be found in [24] pages 82-85.

Proposition 8. The Duflo-Moore operator for the irreducible unitary representation π_+ of $\mathrm{Aff}^+(1)$ as written in eqs. (11) is of the form

$$C_{\pi_+} f(\Delta(x)) = x^{-1/2} f(x), \qquad (f \in \mathrm{L}^2(\mathbb{R}_+),\ x \in \mathbb{R}_+) \tag{15}$$

Proof. Let ϑ_1 and ϑ_2 be elements in $C_c(\mathrm{Aff}^+(1))$. Using the right Haar measure, we shall compute the integral

$$\int_{\mathrm{Aff}^+(1)} \left|\langle \vartheta_1 | \pi_+(\nabla(\beta))\pi_+(\Delta(\alpha))\vartheta_2\rangle_{\mathrm{L}^2(\mathbb{R}_+)}\right|^2 d\beta \frac{d\alpha}{\alpha}$$

To do that, first we compute the following inner product.

$$\begin{aligned}\langle \vartheta_1 | \pi_+(\nabla(\beta))\pi_+(\Delta(\alpha))\vartheta_2\rangle_{\mathrm{L}^2(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} \vartheta_1(x)\overline{\pi_+(\nabla(\beta))\pi_+(\Delta(\alpha))\vartheta_2}\,(x)\ \frac{dx}{x} \\ &= \int_{\mathbb{R}_+} \vartheta_1(x)\overline{\pi_+(\Delta(\alpha))e^{2\pi\iota\beta x}\vartheta_2}\,(x)\ \frac{dx}{x} \\ &= \int_{\mathbb{R}_+} e^{-2\pi i\beta x}\vartheta_1(x)\overline{\pi_+(\Delta(\alpha))\vartheta_2}\,(x)\ \frac{dx}{x}.\end{aligned}$$

Using Plancherel's theorem we have

$$\begin{aligned}\int_{\mathbb{R}} \left|\langle \vartheta_1 | \pi_+(\nabla(\beta))\pi_+(\Delta(\alpha))\vartheta_2\rangle_{\mathrm{L}^2(\mathbb{R}_+)}\right|^2 d\beta &= \int_{\mathbb{R}_+} \left|e^{-2\pi i\beta x}\vartheta_1(x)\overline{\pi_+(\Delta(\alpha))\vartheta_2}(x)\right|^2\ \frac{dx}{x^2} \\ &= \int_{\mathbb{R}_+} \left|\vartheta_1(x)\overline{\pi_+(\Delta(\alpha))\vartheta_2}(x)\right|^2\ \frac{dx}{x^2} \\ &= \int_{\mathbb{R}_+} \left|\vartheta_1(x)\overline{\vartheta_2}(\alpha x)\right|^2\ \frac{dx}{x^2}.\end{aligned}$$

Therefore, using Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathrm{Aff}^+(1)} \left|\langle \vartheta_1 | \pi_+(\nabla(\beta))\pi_+(\Delta(\alpha))\vartheta_2\rangle_{\mathrm{L}^2(\mathbb{R}_+)}\right|^2 d\beta\frac{d\alpha}{\alpha} &= \int_{\mathbb{R}_+} |\vartheta_1(x)|^2 \left\{\int_{\mathbb{R}_+} |\vartheta_2(\alpha x)|^2 \frac{d\alpha}{\alpha}\right\}\frac{dx}{x^2} \\ &= \int_{\mathbb{R}_+} |\vartheta_1(x)|^2 \frac{dx}{x^2}\left\{\int_{\mathbb{R}_+} |\vartheta_2(\alpha')|^2 \frac{d\alpha'}{\alpha'}\right\} \\ &\qquad (\alpha' \coloneqq \alpha x) \\ &= \int_{\mathbb{R}_+} \left|x^{-1/2}\vartheta_1(x)\right|^2 \frac{dx}{x}\left\{\int_{\mathbb{R}_+} |\vartheta_2(\alpha')|^2 \frac{d\alpha'}{\alpha'}\right\} \\ &= \int_{\mathbb{R}_+} \left|x^{-1/2}\vartheta_1(x)\right|^2 \frac{dx}{x}.\ \ \|\vartheta_2\|^2.\end{aligned}$$

Therefore, The Duflo-Moore operator for the irreducible unitary representation of $\mathrm{Aff}^{+}(1)$ as written in eqs. (11) is of the form $C_{\pi_+} f(\Delta(x)) = x^{-1/2} f(x)$ as desired.

■

4. Conclusions

The Duflo-Moore operator for the representations of $\mathrm{Aff}^{+}(1)$ in this paper is considered in two cases. The first case, it is for the quasi-regular representation and written in the term of Fourier transform. Namely, we obtain $\mathcal{F}(C_\pi \psi)(\xi) = \xi^{-1/2} \mathcal{F}\psi(\xi)$ (see [18]**)**. The second case, the Duflo-Moore operator is considered for irreducible unitary representation with respect to its right Haar measure and we have $C_{\pi_+} f(\Delta(x)) = x^{-1/2} f(x)$. On the other hand, the Duflo-Moore operator for a square-intergarble representation of $\mathrm{Aff}^{+}(1)$ with respect to its left Haar measure can be seen in [24] pages 82-85.

It is more interesting to compute the Duflo-Moore operator for the representation of higher dimension of affine Lie groups.

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